Stochastic resonant media: Effect of local and nonlocal coupling in reaction-diffusion models

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We study the phenomenon of stochastic resonance in spatially extended systems or *stochastic resonant media*. Two reaction-diffusion models are analyzed (with one and two components, respectively), both with a known form of the nonequilibrium potential that is exploited to obtain first the probability for the decay of the metastable extended states and second expressions for the correlation function and for the signal-to-noise ratio, within the framework of a two-state description. The analytical results show that this ratio increases with both local and nonlocal coupling parameters. [S1063-651X(98)09505-1]

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I. INTRODUCTION

Since its original proposal as a mechanism accounting for the periodicity in Earth's ice ages [1], the phenomenon of *stochastic resonance* (SR) has been extensively studied from both the theoretical and experimental points of view, as it displays one of the most fascinating cooperative effects arising out of the interplay between deterministic and random dynamics in a nonlinear system. Some recent reviews and conference proceedings clearly show its wide interest and the state of the art [2]. It is worth noting that SR has crossed disciplinary boundaries and its role in sensory and other biological aspects is being explored in diverse experiments [3].

Very recently the attention of researchers has been attracted by the particular features of this phenomenon in the case of coupled or extended systems [4–8], systems that we call *stochastic resonant media* [7]. There are particularly interesting results of numerical simulations of arrays of coupled nonlinear oscillators [5] where it is shown that coupling between first neighbors enhances the response to a weak external periodic signal. In relation to these numerical simulations, we have recently obtained analytical results [7] with a model that corresponds to the continuous limit of the discrete system analyzed in the above-indicated work. In that study, as in a previous one [8], the analysis of this phenomenon in a spatially extended system was made by exploiting previous results [9,10] that were obtained by applying the notion of the so-called *nonequilibrium potential* [11] in reaction-diffusion (RD) models. A related study (also for a one-component system) was done by analyzing the overdamped continuous limit of a ϕ^4 field theory [12], reaching analogous results.

In addition to this problem, which corresponds to a local coupling, and the study of its associated one-component RD model, we have studied the effect of nonlocal couplings as they arise in a two-component RD model of the activator-inhibitor type, in the limit of fast inhibition, where the form of the nonequilibrium potential is also known [13]. The interest of such an analysis is apparent, due to the wide range of applications of these types of models in chemistry, biology, medicine, as well as in technology.

Particularly interesting results ensue from the experiments of Ref. [14]. Those experiments (studies of the Belousov-Zhabotinsky reaction, a peroxidase-oxidase reaction, and a minimum-bromate reaction), done under well-stirred conditions, correspond to transitions between a focus and an oscillatory state via a Hopf bifurcation, where both are homogeneous states. There is also a more recent and also closely related experimental result that corresponds to the case of resonant pattern formation in a chemical system [15]. Even though such cases cannot be described by the activator-inhibitor model in the fast inhibitor limit, they make apparent the relevance of the present results and the interest of further studies exploiting the approach shown here.

The organization of the paper is as follows. In Sec. II we introduce both models and discuss the form of the stationary patterns for different boundary conditions (BCs). We also introduce the form of the nonequilibrium potential (NEP) for both cases. In Sec. III we draw on the results for the local and local plus nonlocal coupling cases in order to study *stochastic resonant media* (SRM) and to show the dependence of the *signal-to-noise ratio* (SNR) on the coupling parameters and its enhancement. To do this we extend and apply the two-level description of bistable systems due to McNamara and Wiesenfeld [16]. Finally, we devote Sec. IV to a general discussion.

II. MODELS OF SPATIALLY EXTENDED SYSTEMS

A. One-component model

We consider first a one-dimensional, one-component model of an electrothermal instability [17,18], which corresponds to an approximation to the continuous limit of the coupled discrete system studied by Lindner *et al.* [5]. In previous studies with this model, we have analyzed the effect of BCs on pattern selection, the *global stability* of nonhomogeneous structures, and the critical-like behavior due to the coalescence of two patterns when a control parameter is varied [9,10].

The bistable RD model that we present here describes the time evolution of a field $\phi(x,t)$, which represents the tem-

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perature profile in the so-called hot spot model in superconducting microbridges [19]. It is also a piecewise linear version of the Schlögl model for an autocatalytic chemical reaction [20] with spatial dependence. The evolution of ϕ is given by

$$\partial_t \phi = D\partial_x^2 \phi - \phi + \theta(\phi - \phi_c) \tag{1}$$

in the bounded domain $x \in [-L,L]$ with Dirichlet or Neumann BCs at both ends, i.e., $\phi(\pm L,t)=0$ or $\partial_x \phi(x,t)|_{\pm L}=0$, respectively, and $\theta(x)$ is the step function. The diffusion constant D is related to the coupling parameter used in the numerical simulations [5].

It is worth remarking here that the present model is also related to the so-called *barreter effect* or *ballast resistor* [18,21,22]. Equation (1) is a dimensionless form of the indicated models, where all the effect of the parameters that keep the system away from equilibrium (for instance, the electric current in the ballast resistor or some reactant concentration in chemical models) are included in ϕ_c.

We also note that since the value of the field $\phi(x,t)$ corresponds in the indicated models to the deviation from a reference temperature [$\phi(x,t)=T(x,t)-T_B$ with $T(x,t)$ the temperature field and $T_B>0$ the temperature of the bath in the ballast resistor] or a reference concentration, it is clear that, up to a certain limit [i.e., $T(x,t)=0$ or $\phi(x,t)=-T_B$ strictly], some negative values of $\phi(x,t)$ are allowed. The relevance of this point will become apparent latter.

The piecewise linear approximation of the reaction term $f(\phi)=-\phi+\theta(\phi-\phi_c)$ mimicking a cubiclike form was chosen in order to find analytical expressions for the spatially symmetric solutions of Eq. (1). It is clear that the trivial solution $\phi_0(x)=0$, which is linearly stable, exists for the whole range of parameters and for both BCs.

For the case of Dirichlet BCs, in addition to the trivial solution, there is only one stable nonhomogeneous structure $\phi_s(x)$, which presents an excited [$\phi_s(x)>\phi_c$] central zone, and another similar unstable structure $\phi_u^D(x)$, with a smaller excited central zone. The latter pattern corresponds to the saddle separating both attractors $\phi_0(x)$ and $\phi_s(x)$ [9,10,18].

For Neumann BCs, there is also only one stable solution in addition to the trivial solution, which is also a homogeneous structure $\phi_1(x)=\phi_1=1$, which represents the excited state ($\phi_1>\phi_c$), while the unstable structure or saddle separating both stable states is a nonhomogeneous one $\phi_u^N(x)$ [18]. The form of the patterns in both cases is depicted in Fig. 1.

These patterns shall be extrema of the NEP or Lyapunov functional of our system. The notion of nonequilibrium potential has been introduced by Graham and Tel [11] and corresponds, loosely speaking, to an extension of the notion of equilibrium thermodynamical potential to nonequilibrium situations. For the present case, it reads [9,10]

$$\mathcal{F}[\phi,\phi_c] = \int_{-L}^{+L} \left\{ -\int_0^{\phi} [-\phi' + \theta(\phi'-\phi_c)]\, d\phi' + \frac{D}{2}(\partial_x \phi)^2 \right\} dx. \tag{2}$$

It can be shown that

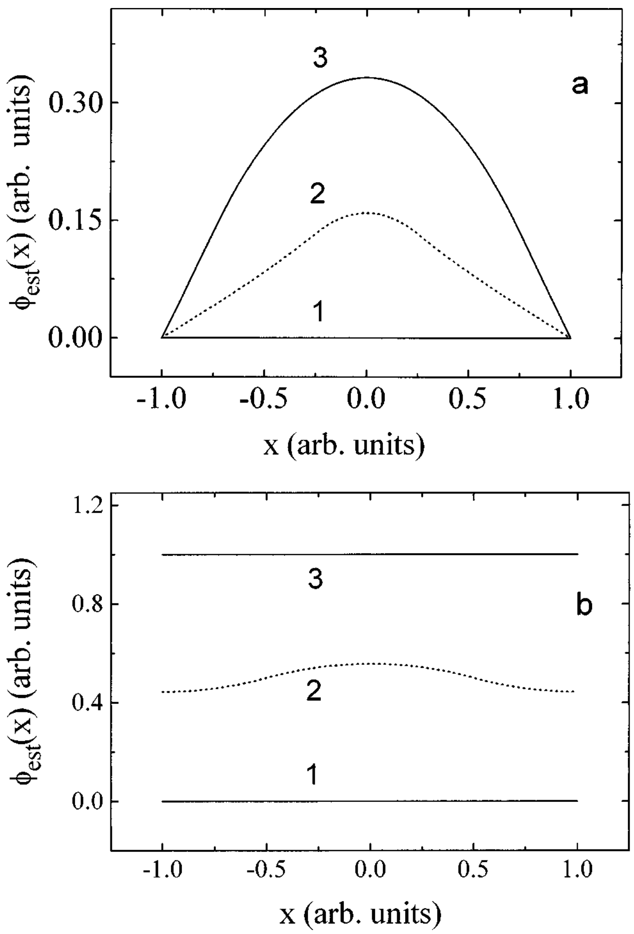

FIG. 1. Form of the patterns for the ballast model for (a) Dirichlet BCs and (b) Neumann BCs. The numbers correspond to (1) the trivial homogeneous solution [$\phi_0(x)$], (2) the saddle [$\phi_u^D(x)$ or $\phi_u^N(x)$], and (3) the nonhomogeneous [$\phi_s(x)$] or (nontrivial) homogeneous (ϕ_1) stable pattern. We adopted $L=1$ and $\phi_c=0.1$

$$\partial_t \phi = -\frac{\delta \mathcal{F}}{\delta \phi} \tag{3}$$

and also that it fulfills

$$\frac{d\mathcal{F}}{dt} = -\int \left(\frac{\delta \mathcal{F}}{\delta \phi}\right)^2 dx \leqslant 0. \tag{4}$$

This Lyapunov functional offers us the possibility to study not only the local, but also the global stability of the patterns, as well as the changes associated with variations of model parameters [9,10].

In Fig. 2 we depict, for Dirichlet BCs, the NEP $\mathcal{F}[\phi,\phi_c]$ evaluated at the stationary patterns ϕ_0 ($\mathcal{F}^0=\mathcal{F}[\phi_0]=0$), $\phi_s(x)$ ($\mathcal{F}^s=\mathcal{F}[\phi_s]$), and $\phi_u(x)$ ($\mathcal{F}^u=\mathcal{F}[\phi_u^D]$), for a system size $L=1$, as a function of ϕ_c for two values of D. The upper branch of each curve is the NEP for $\phi_u^D(x)$, where $\mathcal{F}$ attains an extremum (a saddle). On the lower branch, for $\phi_s(x)$ and also for $\phi_0(x)$, the NEP has local minima. For each value of D the curves exist up to a certain critical value of ϕ_c at which both branches collapse [9,10]. It is interesting to note that since the NEP for $\phi_u^D(x)$ is always positive and, for $\phi_s(x)$, $\mathcal{F}^s$ is positive for some values of ϕ_c and also $\mathcal{F}^s \to -\infty$ as $\phi_c \to 0$, $\mathcal{F}^s$ vanishes for an intermediate value of

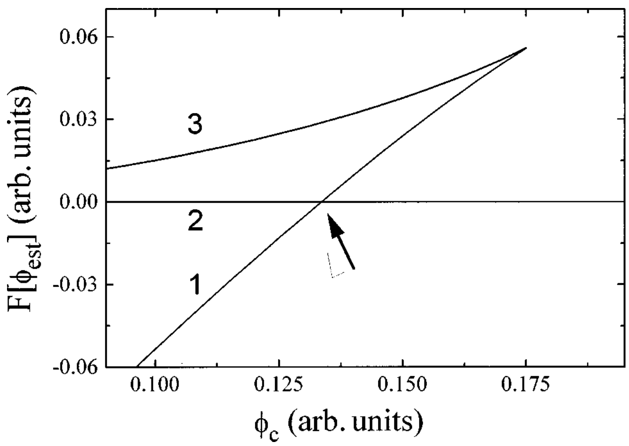

FIG. 2. Nonequilibrium potential $\mathcal{F}$, evaluated at the stationary patterns, for Dirichlet BCs, as a function of ϕ_c for $L=1$ and two values of D: (1) $D=1$ and (2) $D=2$. The bottom curve corresponds to $\phi_s(x)$ and the top one to $\phi_u(x)$. The bistability points ϕ_c^* are indicated.

the threshold parameter $\phi_c=\phi_c^*$, where $\phi_s(x)$ and $\phi_0(x)$ exchange their relative stability.

For the case of Neumann BCs (not shown here), we find that the behavior of $\mathcal{F}[\phi,\phi_c]$ is qualitatively similar to the Dirichlet BC case. The branches corresponding to the evaluation of the NEP on the stationary solutions ϕ_u^N and ϕ_1 have similar shapes, except for the linear dependence of $\mathcal{F}[\phi_1,\phi_c]$ on ϕ_c and the (important) fact that the barrier associated with $\mathcal{F}[\phi_u^N,\phi_c]$ is much larger than in the previous case.

B. Two-component model

The general mathematical formulation for the *activator-inhibitor* model in one spatial dimension reads [21–23]

$$\partial_t u = D_u \partial_x^2 u + f(u,v),$$

$$\partial_t v = D_v \partial_x^2 v + g(u,v), \tag{5}$$

where D_u and D_v are the diffusion coefficients of the activator $u(x,t)$ and the inhibitor $v(x,t)$, respectively. Both $u(x,t)$ and $v(x,t)$ are real fields representing the magnitudes of interest and the nonlinear terms $f(u,v)$ and $g(u,v)$ are the reaction terms.

The null clines, which are the intersections of those (generally nonlinear) source terms with the (u,v) plane, show characteristic shapes that can be typically described by a convex line for $g(u,v)$ and a general cubiclike one (with two extrema and one inflection point) for $f(u,v)$ [21–24]. Those projections intersect each other at the origin (which counts for a trivial solution) and eventually on both sides around the local maximum of f. Those extra intersections anticipate nontrivial homogeneous solutions and spatial patterns arising from the bistable situation of the system, which have been analyzed by several authors [21,22,24–26]. It is worth remarking here that, similarly to what happens with the previous one-component model, the fields $u(x,t)$ and $v(x,t)$, for instance, in the case of a chemical system, describe the deviation of some reactant density from a reference value and hence some negative values are also allowed [21–23]. As in the previous case, we will work with a simplified (piecewise linear) version of the activator-inhibitor model alluded to above, which preserves the essential features, and fix the parameters so as to allow nontrivial solutions to exist. After scaling the fields in a standard way, we get a dimensionless version of the model as

$$\partial_t u(x,t) = D_u \partial_x^2 u - u + \theta(u-u_c) - v,$$

$$\partial_t v(x,t) = D_v \partial_x^2 v + \beta u - \eta v. \tag{6}$$

We again confine the system to the interval $-L<x<L$ and restrict ourselves to imposing Dirichlet BCs in both extrema $[u(\pm L)=v(\pm L)=0]$ (Neumann BCs, as in the previous case, are less interesting). Depending on the values of the parameters u_c, β, and η, we can have a monostable (excitable) or a bistable situation as qualitatively indicated in Ref. [13]. In the second case we have two homogeneous stationary (stable) solutions. One corresponds, in the (u,v) plane, to the point (0,0), while the other is given by (u^0,v^0), with

$$u^0 = \frac{\eta}{\beta+\eta}, \quad v^0 = \frac{\beta}{\beta+\eta},$$

implying that the condition $\eta/(\beta+\eta)>u_c$ must be fulfilled. Without losing generality, we may assume that $0<u_c<1/2$ and $u_0<2u_c$ [24]. We choose values of the parameters β and η so as to work in the excitable regime.

The inhomogeneous stationary patterns appear due to the nonlinearity of the system and ought to have activated regions ($u>u_c$) coexisting with nonactivated regions ($u<u_c$). This fact, together with the symmetry of the evolution equations and BCs, implies the existence of symmetric inhomogeneous stationary solutions. We restrict ourselves to the simplest inhomogeneous, symmetric, stationary solutions, that is, a symmetric pattern consisting of a central region where the activator field is above a certain threshold ($u>u_c$) and two lateral regions where it is below it ($u<u_c$). As was already discussed [22,24], different analytical forms (which are here linear combinations of hyperbolic functions) should be proposed for u and v depending on whether $u>u_c$ or $u<u_c$. These forms, as well as their first derivatives, need to be matched at the spatial location of the transition point, which we called x_c. Through that matching procedure and imposing boundary conditions we get the general solution for the stationary case. In order to identify the matching point x_c we have to solve the equation $u(x_c)=u_c$, leading in general to a transcendental equation for x_c [27].

We have analyzed a restricted parameter region in order to have only two different solutions for x_c and associated with each we have a stationary solution that we will indicate by u_u and u_s, for increasing values of the transition point x_c [28,29]. The shape of the patterns, at least for the activator profile, is analogous to those found for the one-component case (see Fig. 1). A linear stability analysis of these solutions indicates that u_u is unstable while u_s is locally stable. As before, the stable states shall correspond to attractors (minima) of the NEP while the unstable ones are linked to saddles, defining the barrier height between attractors. In addition to these two patterns, we always have the trivial (stable) homogeneous solution $u^0 = v^0 = 0$.

We write now the equations of our system specifying the time scale associated with each field. We can then perform an adiabatic approximation and obtain a particular form of the NEP for this system. Measuring the time variable on the characteristic time scale of the slow variable u (i.e., τ_u), Eqs. (6) adopt the form [21–23]

$$\partial_t u(x,t) = D_u \partial_x^2 u(x,t) - u(x,t) + \theta(u(x,t) - u_c) - v(x,t),$$

$$\zeta \partial_t v(x,t) = D_v \partial_x^2 v(x,t) + \beta u(x,t) - \eta v(x,t), \qquad (7)$$

where $\zeta = \tau_v / \tau_u$ [31]. At this point we assume that the inhibitor is much faster than the activator (i.e., $\tau_v \ll \tau_u$). In the limit $\zeta \to 0$, we can rewrite Eqs. (7) as

$$\partial_t u(x,t) = D_u \partial_x^2 u(x,t) - u(x,t) + Z\theta(u(x,t) - u_c) - v(x,t),$$

$$0 = D_v \partial_x^2 v(x,t) + \beta u(x,t) - \eta v(x,t). \qquad (8)$$

We can eliminate the inhibitor (which is *slaved* to the activator) by solving the second equation using the Green's-function method

$$\{-D_v \partial_x^2 + \eta\} G(x,x') = \delta(x - x'),$$

$$v(x) = \beta \int dx' G(x,x') u(x'), \qquad (9)$$

where the Green's function $G(x,x')$ is given by

$$G(x,x') = \begin{cases} \dfrac{1}{D_v k} \dfrac{\sinh[k(L-x')]}{\sinh[2kL]} \sinh[k(L+x)], & x < x' \\ \dfrac{1}{D_v k} \dfrac{\sinh[k(L-x)]}{\sinh[2kL]} \sinh[k(L+x')], & x > x', \end{cases} \qquad (10)$$

with $k = (\eta / D_v)^{1/2}$. This slaving procedure reduces our system to a *nonlocal* equation for the activator only, which has the form

$$\partial_t u(x,t) = D_u \partial_x^2 u(x,t) - u(x,t) + \theta(u(x,t) - u_c)$$

$$- \beta \int dx' G(x,x') u(x'). \qquad (11)$$

From this equation and taking the symmetry of the Green's function $G(x,x')$ into account, we can obtain the NEP for this system (which we will indicate by $\mathcal{H}[u,u_c]$ to avoid confusion with the previous case), which allows us to obtain

$$\partial_t u(x,t) = - \frac{\delta \mathcal{H}[u,u_c]}{\delta u}, \qquad (12)$$

where $\mathcal{H}[u,u_c]$ has the form

$$\mathcal{H}(u) = \int dx \left\{ \frac{D_u}{2} \{\partial_x u\}^2 + \frac{u^2}{2} - (u - u_c)\Theta(u - u_c) + \frac{\beta}{2} \int dx' G(x,x') u(x') u(x) \right\}$$

$$= \mathcal{F}[u,u_c] + \frac{\beta}{2} \int dx' G(x,x') u(x') u(x). \qquad (13)$$

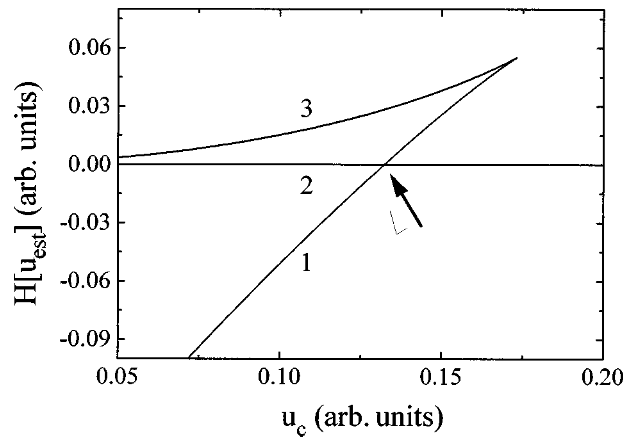

FIG. 3. Nonequilibrium potential $\mathcal{H}$ for the stationary patterns, for Dirichlet BCs, as a function of u_c for $L=1$, $D_u=1$, $D_v=4$, $\eta = 0.08$, and $\beta = 0.4$. The bottom curve corresponds to $u_s(x)$ and the top one to $u_u(x)$. The bistability point u_c^* is indicated.

Here $\mathcal{F}[u,u_c]$ has the same functional form as in Eq. (2), replacing ϕ by u and ϕ_c by u_c. Clearly, $\mathcal{H}$ also fulfills the condition $d\mathcal{H}/dt \leq 0$. The spatial nonlocal term in the NEP takes into account the repulsion between activated zones. When two activated zones approach each other, the exponential tails of the inhibitor concentration overlap, increasing its concentration between both activated zones and creating an *effective* repulsion between them, the Green's function playing the role of an exponential screening between the activated zones.

In Fig. 3 we show the dependence of $\mathcal{H}$ on u_c for the different stationary patterns, with $L=1$ as in the previous case. The upper branch corresponds to the NEP evaluated on the unstable pattern u_u ($\mathcal{H}^u$), while the lower one corresponds to the NEP evaluated on the stable (nonhomogeneous) pattern u_s ($\mathcal{H}^s$). For the trivial homogeneous solution u_0 we have $\mathcal{H}^0 = \mathcal{H}[u_0,u_c] \equiv 0$. As for the local coupling case, the lower branch $\mathcal{H}^s$ is negative for small u_c and becomes positive for larger values. At some intermediate point $u_c = u_c^*$, we find that $\mathcal{H}^s = \mathcal{H}[u_s, u_c^*] = \mathcal{H}^0 = 0$. Hence, at this point the locally stable patterns u_s and u_0 exchange their relative stabilities. Also, in the present case, as in the previous one, for some value of u_c (larger than u_c^*) both branches u_u and u_s coalesce and dissappear at a critical point [10].

III. STOCHASTIC RESONANT MEDIA

A. Local coupling

In order to study SR in an extended system or SRM, we must introduce an external noise source and a weak signal that modulates the potential $\mathcal{F}$ about the situation in which the two wells (representing the two stable states) have the same depth. This is accomplished by letting the parameter ϕ_c oscillate around ϕ_c^*,

$$\phi_c(t) = \phi_c^* + \delta\phi_c \cos(\Omega t + \varphi), \qquad (14)$$

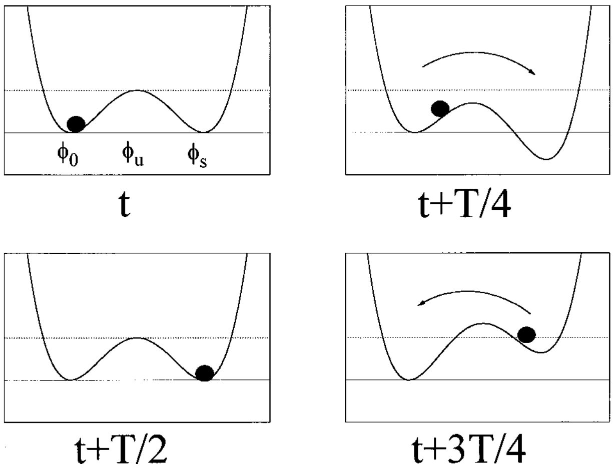

FIG. 4. Qualitative (one dimensional) behavior for the nonequilibrium potential under the effect of an external modulation for the ballast model. Here ϕ_0, ϕ_u, and ϕ_s indicate the ''points'' of the trivial homogeneous solution, the unstable nonhomogeneous solution, and the stable nonhomogeneous solution, respectively. Also, t indicates an arbitrary initial time, while T corresponds to the modulation period.

where φ is an arbitrary (random) initial phase. We assume that the amplitude of the modulation ($\delta\phi_c$) is small enough in order not to destroy the bistability of the system. It is worth remarking here that the present situation shows some differences from the usual bistable systems. Here the state ϕ_0 has always $\mathcal{F}^0=0$, while what changes from being metastable to stable (or vice versa) is the state $\phi_s(x)$, that is, $\mathcal{F}^s$ lies above or below $\mathcal{F}^0=0$, depending on whether ϕ_c is larger or smaller than ϕ_c^*. Also, the barrier associated with $\phi_u(x)$, i.e., $\mathcal{F}^u$, increases or reduces accordingly. This behavior is qualitatively depicted in Fig. 4.

In order to account for the effect of fluctuations, we include in the time evolution equation of our model [Eq. (1)] a fluctuating term $\xi(x,t)$, which we model as an additive Gaussian white noise source with zero mean value and a correlation function $\langle \xi(x,t)\xi(x',t')\rangle = 2\gamma\delta(t-t')\delta(x-x')$, yielding a stochastic partial differential equation for the random field $\phi(x,t)$. The parameter γ denotes the noise strength.

Strictly speaking, the fluctuations cannot be Gaussian, but must have an absorbing boundary at the lower limit [i.e., $T=0<T_B$ or $\phi(x,t)=-T_B<0$ for the case of the ballast resistor] where the noise has to vanish. However, we expect that the differences introduced by the Gaussian negative tail beyond the indicated absorbing boundary will not drastically change our results (this is in fact justified by the qualitative agreement with the simulations of Ref. [5]). For this reason and in order to simplify the analysis, in what follows we assume that fluctuations are Gaussian.

When studying zero-dimensional or nonextended systems the usual methodology has been to estimate the decaying time by means of Kramers-like formulas. However, there have been a few attempts to go beyond such an approximation (which, as a matter of fact, loses its validity when the size of the fluctuations becomes of the same order as the barrier height), for instance, through a Floquet analysis [30] to study such a nonstationary situation. Clearly, for extended

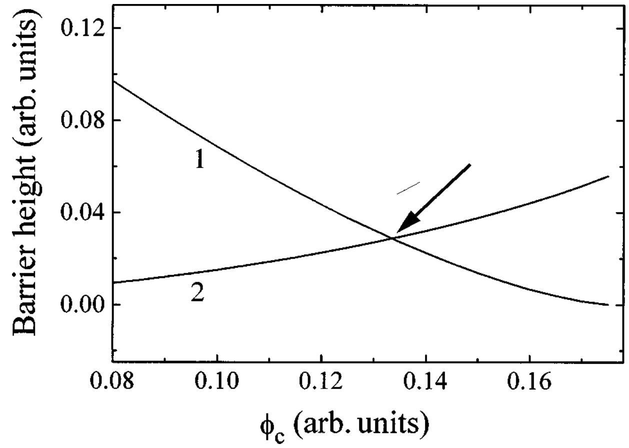

FIG. 5. Behavior, for the ballast model, of the barrier height as a function of ϕ_c: 1 corresponds to the barrier for $\phi_s(x)$ ($\Delta\mathcal{F}[\phi_s,\phi_c^*]$) and 2 the barrier for $\phi_0(x)$ ($\Delta\mathcal{F}[\phi_0,\phi_c^*]$). We took $L=1$ and $D=1$.

systems such a kind of analysis is far from trivial (and beyond the scope of the present study) as we need to consider a functional (or infinite-dimensional) Fokker-Planck equation. Hence we adopt a viewpoint analogous to that used in zero-dimensional systems, that is, to look for extensions of the Kramers approach to the present situation.

At this point we exploit a scheme, based on path-integral techniques, that allows us to describe the decay of extended metastable states [32], yielding the following Kramers-like result for the decaying time or first-passage time $\langle\tau\rangle$:

$$\langle\tau\rangle = \tau_0 \exp\left\{\frac{\Delta\mathcal{F}[\phi,\phi_c]}{\gamma}\right\}, \tag{15}$$

where

$$\Delta\mathcal{F}[\phi,\phi_c] = \mathcal{F}[\phi_{unst}(y),\phi_c] - \mathcal{F}[\phi_{meta}(y),\phi_c]. \tag{16}$$

The prefactor τ_0 is determined by the curvature of $\mathcal{F}[\phi,\phi_c]$ at its extrema. The range of validity of this expression is analogous to that of the classical Kramers formula. As the approximation essentially rests on extracting the contribution from the most probable trajectory, its validity is also restricted to a region where the noise intensity is much smaller than the barrier height. Due to the difficulties in improving such an evaluation of the decay time, as in the case of nonextended systems [1,7,8,12,16] we will continue using this expression even beyond its validity range. In Fig. 5 we show the form of $\Delta\mathcal{F}[\phi_0,\phi_c]$ (line 1) and $\Delta\mathcal{F}[\phi_s,\phi_c]$ (line 2) as a function of ϕ_c. Clearly, these curves are also related to the behavior of $\ln(\langle\tau\rangle/\tau_0)$. For the spatially extended problem, we need to evaluate the space-time correlation function $\langle\phi(y,t)\phi(y',t')\rangle$ and to make a double Fourier transform of the correlation function in order to obtain, instead of the power spectrum, the *generalized susceptibility* $S(\kappa,\omega)$ [22].

To proceed with the calculation of the correlation function we use a simplified point of view (due to the bistable character of our problem potential), based on the two-state approach of McNamara and Wiesenfeld [16] and described in the Appendix, which allows us to apply some of their results almost straightforwardly. We restrict the description by assuming that the relaxation of the system inside each basin of attraction towards the corresponding attractor is much faster than the typical transition time between attractors (a study of the linear stability eigenvalues indicates that this is the case if Ω is small enough). Hence we shall concentrate on the transitions between both stationary states ϕ_0 and ϕ_s. To calculate the correlation function we need to evaluate the transition probabilities between our two states ϕ_0 and ϕ_s, which appear in the associated master equation

$$W_{0,s}=\tau_0^{-1}\exp(-\Delta\mathcal{F}^{0,s}[\phi_{0,s},\phi_c]/\gamma), \tag{17}$$

where τ_0, an estimation of the curvature at the potential extrema, is given by the asymptotically dominant linear stability eigenvalues

$$\tau_0=\frac{2\pi}{\sqrt{|\lambda^{un}|\overline{\lambda^{st}}}}$$

(λ^{un} is the only unstable eigenvalue around ϕ_u and $\overline{\lambda^{st}}$ is the average of the minimum eigenvalues around ϕ_0 and ϕ_s) and for small $\delta\phi_c$,

$$\Delta\mathcal{F}[\phi,\phi_c]\approx\Delta\mathcal{F}[\phi,\phi_c^*]+\delta\phi_c\left[\frac{\partial\Delta\mathcal{F}[\phi,\phi_c]}{\partial\phi_c}\right]_{\phi_c^*}\times\cos(\Omega t+\varphi). \tag{18}$$

In analogy to the nonextended case, we have assumed here that, as we will only work in the limit of very small and very slow modulation (that is, both $\delta\phi_c$ and Ω small enough), the nonstationary transition probability can be perturbatively expanded in $\delta\phi_c$, as we only require the solution of the master equation up to first order in this parameter. Solving the master equation, it is possible to evaluate the correlation function and to perform its Fourier transform in time as well as in space in order to obtain the generalized susceptibility $S(\kappa,\omega)$. These expressions are similar to those obtained by McNamara and Wiesenfeld [16] (see the Appendix). Using a by now standard definition for the SNR, we have shown in the Appendix that the relevant part of the SNR is given by

$$R\sim\left(\frac{\Lambda}{\tau_0\gamma}\right)^2\exp(-2\Delta\mathcal{F}[\phi,\phi_c^*]/\gamma), \tag{19}$$

with

$$\Lambda=\left[\frac{d\Delta\mathcal{F}}{d\phi_c}\right]_{\phi_c^*}\delta\phi_c. \tag{20}$$

Equation (19) is analogous to what has been obtained in zero-dimensional systems, but where Λ, τ_0, and $\Delta\mathcal{F}[\phi,\phi_c^*]$ contain all the relevant information regarding the spatially extended character of the system.

In Fig. 6 we show the dependence of the present SNR on γ for typical values of the parameters (same as in the previous figures) and different values of D. We can see that the response increases for increasing values of D. In Fig. 7 we show the dependence of the SNR (for fixed noise) as a function of D that, according to the continuous limit of the system studied in Ref. [5], plays the role of the coupling parameter. This result is in excellent qualitative agreement with the recent numerical results for a system of coupled nonlinear overdamped oscillators [5].

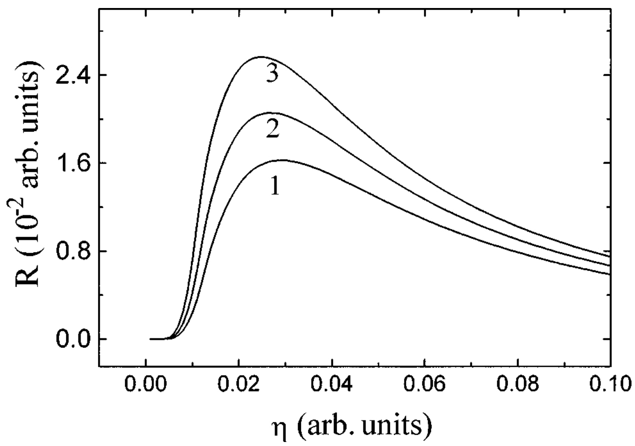

FIG. 6. SNR for the local coupling case, as a function of the noise intensity γ [Eq. (17)], for (1) $D=0.9$, (2) $=1.0$, and (3) $=1.1$. We fixed $\phi_c=\phi_c^*$, $L=1$, $\delta\phi_c=0.01$, and $\Omega=0.01$.

B. Local plus nonlocal coupling

Here we proceed as in the previous case and consider that the modulation is introduced through the activator threshold

$$u_c(t)=u_c^*+\delta u_c\cos(\Omega t+\varphi), \tag{21}$$

where u_c^* (as ϕ_c^* before) is the threshold value at which $\mathcal{H}[u_s(x),u_c^*]\equiv 0$. The choice of the parameters was such that we have a bistable situation (only two attractors) in order to be able to exploit the same approach indicated before [32]. We assume that the noise only enters (additively) into the activator equation [33].

The main difference from the previous case arises from the extra (nonlocal) term in $\mathcal{H}[u,u_c]$ when compared with $\mathcal{F}[\phi,\phi_c]$ [see Eq. (13)]. Hence, in addition to the dependence on D_u (the analysis indicates that the dependence on D_v is negligible), we also have the dependence on the pa-

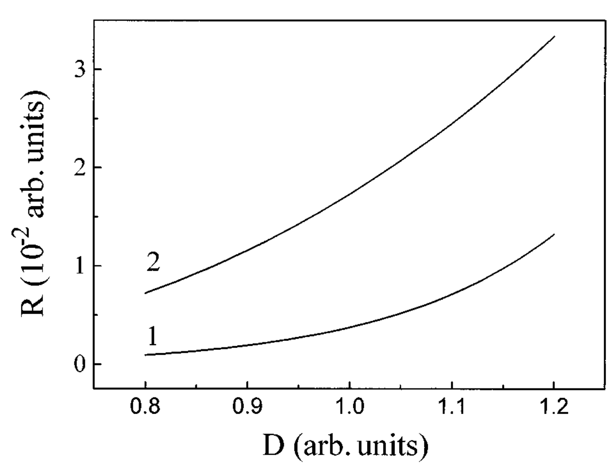

FIG. 7. For the local coupling case, the SNR as a function of D for the same parameters as in Fig. 6 and two values of the noise intensity: (1) $\gamma=0.02$ and (2) $\gamma=0.01$.

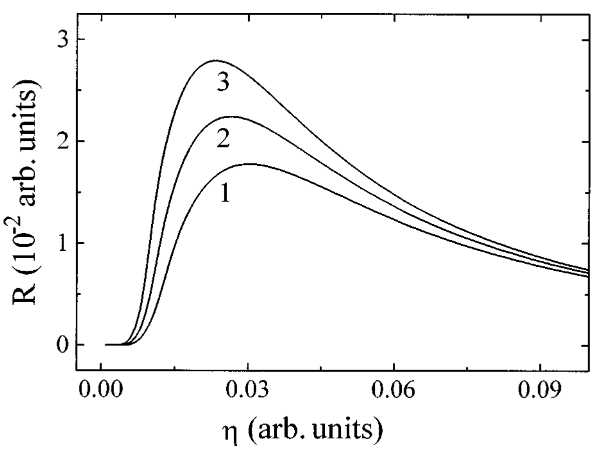

FIG. 8. SNR for the nonlocal coupling case, as a function of the noise intensity γ [analogous to Eq. (17)], for (1) $D_u = 0.9$, (2) $=1.0$, and (3) $=1.1$. We fixed $u_c = u_c^*$, $L=1$, $\delta a = 0.01$, and $\Omega = 0.01$. The rest of the parameters are as in Fig. 3.

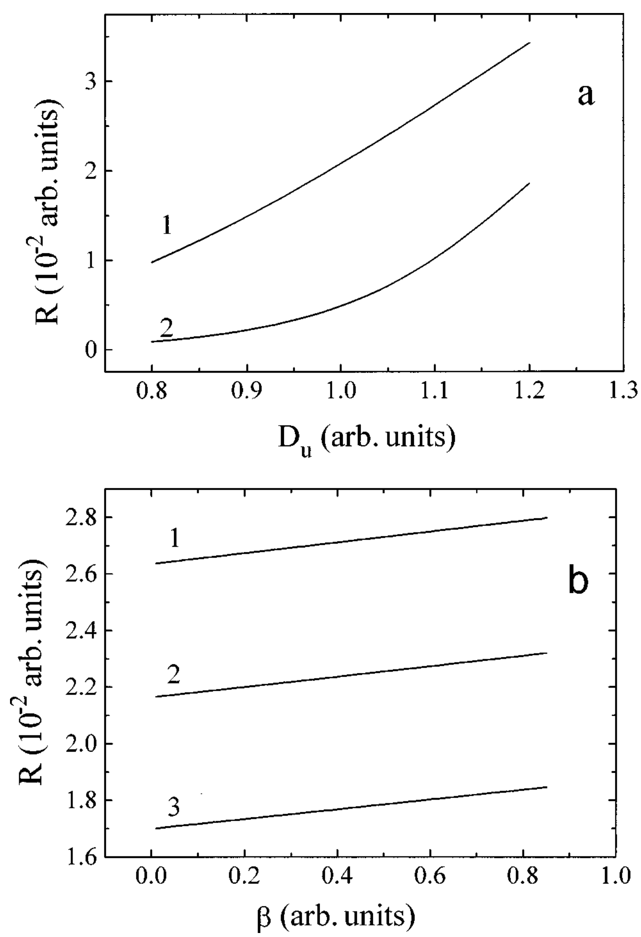

FIG. 9. SNR for the nonlocal coupling case, (a) as a function of D_u for two values of the noise intensity: (1) $\gamma = 0.02$ and (2) $\gamma = 0.01$ and (b) as a function of β for different values of D_u: (1) $D_u = 1.1$, (2) $=1$, and (3) $=0.9$. The rest of the parameters are as in Fig. 8.

rameter β, which measures the strength of the nonlocal coupling. In the limit of $\beta \to 0$, we exactly recover the case with only a local interaction.

It is clear that in the spatial bistable case, coming from the original excitable regime we have analyzed, we will obtain the same kind of result for the first-passage time $\langle \tau \rangle$, that is, an equation similar to Eq. (15) but replacing $\Delta\mathcal{F}$ by $\Delta\mathcal{H}$ [with a definition analogous to Eq. (16)]. Using again the two-state approach [16], we will get expressions for the transition probabilities $W_{0,s}$ similar to the ones indicated in Eq. (17). For small δu_c we will obtain

$$\Delta\mathcal{H}[u,u_c] \approx \Delta\mathcal{H}[u,u_c^*] + \delta a \left[\frac{\partial \Delta\mathcal{H}[u,u_c]}{\partial u_c} \right]_{u_c^*} \cos(\Omega t + \varphi). \tag{22}$$

All these steps will lead us, in the present case, to an expression for the SNR analogous to the one in Eq. (19), where $\Delta\mathcal{F}$ is replaced by $\Delta\mathcal{H}$ and τ_0, Λ are similar quantities (an estimation of the curvature at the potential extrema and $\Lambda = [d\Delta\mathcal{H}/du_c]_{u_c^*}\delta u_c$, respectively). In Fig. 8 we show the dependence of the SNR on the noise intensity for three values of D_u and a fixed β. Figure 9(a) depicts the dependence of the SNR (for fixed noise and β) on D_u, while Fig. 9(b) does the same (for fixed noise and D_u) on β. The enhancement of the SNR with increasing D_u and/or β is apparent from these figures.

IV. DISCUSSION

In order to study the phenomenon of SR in coupled or extended systems or SRM (with the aim to encourage experimentalists dealing with distributed electronic, chemical, or biological systems to search for alternative variables to tune up so as to enhance the stochastic resonant response of the system) we have analyzed two models corresponding to the cases of local and local plus nonlocal coupling. These models are associated with a bistable monocomponent and an activator-inhibitor RD system, respectively. The analysis was done by exploiting the knowledge of the form of the nonequilibrium potential.

The case of local coupling corresponds to the continuous limit of the discrete model discussed by Lindner *et al.* [5], which is associated with a coupled set of nonlinear oscillators in the overdamped limit. We have shown that, in agreement with those simulations, the present results indicate an enhancement of the SNR as a function of the diffusion constant D that plays the role of the coupling parameter. Such an effect is more remarkable for Dirichlet BCs than for Neumann BCs. This difference can be attributed, according to Eq. (19), to the fact that the size of the potential barrier separating both attractors is larger for the latter than in the former case. This difference in the barrier size can be interpreted as follows. For Neumann BCs, the stable stationary patterns have a homogeneous structure indicating a more ''rigid'' behavior when subject to fluctuations, implying that the transitions will be similar to the uncoupled ($D=0$) case, a situation that can be easily evaluated [29]. On the other hand, for Dirichlet BCs, the nonhomogeneity of one of the stable stationary patterns (with part of the pattern below the threshold ϕ_c) makes it easier for the fluctuations to induce a transition between both attractors. This interpretation, which at first sight seems to be restricted to the present piecewise linear model, can also be extended to more general bistable models [29].

It is worth remarking here that the present calculation breaks down for large values of D. This is because, for increasing D, the curves in Fig. 2 shift to the left while the barrier separating the attractors tends to zero, making invalid the applicability of Eq. (19). We see from Eq. (1) that this limit corresponds to diffusion in a monostable potential.

Regarding the activator-inhibitor case, the present results, in addition to the enhancement due to the local coupling, also show an enhancement of the system's response with the nonlocal coupling parameter. The main contribution to this effect again comes from the reduction of the potential barrier when this parameter increases. It is clear that this system corresponds to a more interesting and useful case of a nonlinear oscillator than for the local coupling case, describing not only a pure bistable but also an excitable (and more realistic) situation. We must remark here that, as in the local coupling case, the present form of calculation for the nonlocal coupling breaks down for large values of D_u or β for the same reasons as before. The extension of the present form of analysis to a full (nonslaved) version of the activator-inhibitor case is under way [34].

The relevance of these results for technological applications in signal detection as well as its biological implications are apparent [35–40]. Many distributed electronic circuits can be regarded in the continuum limit as a set of diffusively coupled nonlinear oscillators. With regard to chemical systems, in addition to the particularly interesting results in experiments on several reactions done under well-stirred conditions [14], there is a more recent and also closely related experimental result that corresponds to the case of resonant pattern formation in a chemical system [15], indicating the possibility of the appearance of SR under nonstirred conditions. Even though such cases cannot be described by the activator-inhibitor model in the fast inhibitor limit, they make apparent the relevance of such results and the interest of further studies exploiting the approach shown here. Since the present results *predict* a strong dependence of the SR upon both spatial and interspecies coupling parameters, we hope that they can motivate not only new simulations of coupled sets of such nonlinear oscillators (in the spirit of the numerical analysis of Ref. [5]), but also the experimental search of this spatially dependent phenomenon in chemical and coupled electronic systems. In particular, we expect that by exploiting an experimental setup similar to the one in Ref. [15], with a low-amplitude (below threshold) forcing plus noise, a SRM phenomenon will show up.

Finally, it is worth noting that, in addition to the approximation involved in the Kramers-like expression in Eq. (15) and the two-level approximation used for the evaluation of the correlation function, all the previous results (form of the patterns and nonequilibrium potential) are analytically exact. However, in a more careful analysis of the problem we can expect different strengths for the SR phenomena for different wavelengths that could lead to some kind of spatiotemporal synchronization phenomenon. The dependence of the generalized susceptibility $S(\kappa,\omega)$ on κ and ω, which will not necessarily factorize, will also imply that $R_{\mathrm{SN}}\sim R_{\mathrm{SN}}(\kappa,\omega)$. These aspects are currently under study.

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APPENDIX

According to the theory of McNamara and Wiesenfeld [16], the bistable case is reduced to a two-state system, characterized by the occupation probabilities of both states n_+ and n_- for the (symmetric) states $-c$ and c, respectively (with $n_+ + n_- = 1$). The master equation for these occupation probabilities is

$$\dot{n}_+(t) = W_-(t)n_-(t) - W_+(t)n_+(t), \tag{A1}$$

where $W_\pm(t)$ are the (time-dependent) transition probabilities from the right (c) to the left $(-c)$ well and vice versa, respectively. The general solution solution of Eq. (A1) is

$$n_+(t) = g(t)^{-1}\left[n_+(t_0)g(t_0) + \int_{t_0}^{t} dt' W_-(t')g(t')\right], \tag{A2}$$

with $g(t) = \exp\{\int_0^t dt'[W_-(t') + W_+(t')]\}$.

In Ref. [16], to the lowest order in the amplitude of the modulation, the adopted form of the time-dependent transition probabilities was [see Eq. (3.7) in [16]]

$$W_\pm(t) = f(\mu \pm \epsilon \cos(\Omega t)) \approx \frac{1}{2}[\alpha_0 \mp \alpha_1 \epsilon \cos(\Omega t)] + O(\epsilon^2), \tag{A3}$$

with ϵ a smallness parameter (proportional to the modulation amplitude) and

$$\alpha_0 = 2f(\epsilon = 0),$$

$$\alpha_1 = -2\frac{df}{d\epsilon}\bigg|_{\epsilon=0},$$

where $f(\mu)$ is essentially given by the inverse of the Kramers time. Hence all the information about the transition probabilities is contained in α_0 and α_1.

The result for the power spectrum [see Eqs. (3.12) and (3.13) in [16]] was

$$S(\omega) = \left[1 - \frac{\alpha_1^2\epsilon^2}{2(\alpha_0^2+\Omega^2)}\right]\left[\frac{2\alpha_0 c^2}{(\alpha_0^2+\omega^2)}\right] + \frac{\pi c^2\alpha_1^2\epsilon^2}{2(\alpha_0^2+\Omega^2)}[\delta(\omega-\Omega)+\delta(\omega+\Omega)]. \tag{A4}$$

From this power spectrum the SNR results

$$R(\alpha_0,\alpha_1) \approx \frac{\pi\alpha_1^2\epsilon^2}{4\alpha_0}\left[1 - \frac{\alpha_1^2\epsilon^2}{2(\alpha_0^2+\Omega^2)}\right]^{-1}. \tag{A5}$$

These results, obtained for the symmetric case, can be inmediately adapted for the nonsymmetric case. Assume we have the minima at c_1 and c_2 instead of at $\pm c$. Hence, changing to new coordinates defined by $x' = ax + b$, with a

$=(c_2-c_1)/2$ and $b=(c_2+c_1)/2$, the correlation function will be multiplied by a^2 and we must replace c^2 by $[c_2-c_1/2]^2$ in the power spectrum and similarly in the SNR.

In our case (say, the local coupling one) we have reduced our problem to the transitions between two states $\phi_0\equiv 0$ and $\phi_s(x)$, in the neighborhood of the bistable point $\phi_c=\phi_c^*$. With the above-indicated change in the nonsymmetric bistable case and taking into account the forms of $W_{0,s}$ indicated by Eqs. (17) and (18), we can identify the actual forms of α_0 and α_1, yielding an expression for the correlation function similar to Eq. (3.12) in Ref. [16], where we shall identify c_1 by $\phi_0(x)$ and c_2 by $\phi_s(x)$, *yielding an essentially spatially independent SNR*. Its double Fourier transform yields the generalized susceptibility given by

$$S(\kappa,\omega)=F(\kappa)S(\omega), \tag{A6}$$

where $F(\kappa)\sim\delta(\kappa)$, κ and ω being the Fourier conjugate variables to the space and time ones. $S(\omega)$, the usual power spectrum function (function only of ω), again becomes the relevant quantity. Finally, the relevant contribution for the SNR is the one indicated in Eq. (19) with Λ given by Eq. (20).

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